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A Space-time Functional Formalism for Turbulence

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ABSTRACT

The statistical theory of turbulence is formulated in terms of the characteristic functional Γ of a probability distribution of velocity fields in space-time. A functional differential equation for Γ is derived from the Navier-Stokes equation and the condition of incompressibility. All the moments of the velocity field may be determined as functional derivatives of Γ . The n-point space-time distribution functions of the velocity field ($n=1,2,\dots$) are directly determined from the values of Γ for particular argument-functions. The present theory generalizes that of E. Hopf in that non-simultaneous as well as simultaneous correlation functions are obtained from Γ . In addition, external driving forces are included in the formalism. This permits the representation, as functional derivatives of Γ , of the Green's functions which describe the averaged response of the velocity field to infinitesimal perturbations. As an illustration of the formalism, an explicit solution of the functional equation for Γ is obtained in the limiting case of vanishing Reynolds number, and the known results of the theory of "weak turbulence" are thereby recovered.

1. Introduction and Summary

A functional formalism for turbulence which evolves in time according to the Navier-Stokes equation has been given by Hopf, [2,3] who introduced the characteristic functional Φ of a probability distribution of simultaneous velocity amplitudes and derived a functional differential equation for Φ . If the solution of this equation can be determined, one can obtain from it all correlation functions (moments) of the velocity field which involve simultaneous time-arguments.

In the present paper, we give a generalization of Hopf's formalism in which the central role is played by a characteristic functional Γ of the full space-time distribution of the velocity amplitudes. This permits the determination of the non-simultaneous as well as the simultaneous correlation functions. As a further generalization, we include driving forces in the Navier-Stokes equation. In Section 2, we define Γ and show how the correlation functions are obtained from Γ by functional differentiation. We also show how the n-point space-time probability distributions of the velocity field ($n=1,2,\dots$) can be expressed in terms of Γ .

In Section 3, we derive a functional differential equation for Γ from the Navier-Stokes equation and the incompressibility condition. The formalism is presented first in terms of a space-time representation and then is transformed, in Section 4, into a wave vector-time representation. As an unexpected by-product of our attempt to generalize Hopf's work, we find that the new formalism is somewhat simpler, in that certain complications associated with the elimination of the pressure term from the equations are avoided.

The application of the formalism is illustrated in Section 5, where

we obtain an explicit solution of the functional differential equation in the limiting case where the characteristic Reynolds number of the turbulence vanishes and the Navier-Stokes equation becomes linear. The solution is obtained by analogy to elementary methods of solving linear ordinary differential equations. From our solution we recover the known results of the theory of "weak turbulence".

In Section 6, we express, in terms of Γ , a sequence of Green's functions which describe statistically the response of the velocity field to infinitesimal perturbations in the driving forces. As an illustration, we evaluate the Green's functions in the zero Reynolds number limit.

The analysis outlined above is restricted to the case of non-random driving forces. The extension to stochastic driving forces is given in Section 7. We conclude the paper with a brief description of how the conditions of statistical homogeneity appear in our formalism (Section 8).

The motivation of the present paper is essentially that of Hopf's original paper: It is hoped that the compact expression for the dynamical equations which the functional formalism affords may suggest novel approximation schemes that would not be apparent from the equivalent infinite set of moment equations. So far this hope has not been realized. The obvious procedures suggested by the functional formalism are expansion in moments and expansion in cumulants. Both of these procedures are also obvious from a moment formulation. We hope, however, that the present enlargement of the functional formalism to include the full space-time distribution and the Green's functions of the velocity field may provide a context in which new approximations can be developed.

2. Characteristic Functional and Space-time Correlation Functions

The statistical theory of turbulence may be described in terms of a probability distribution P in the space of velocity vector fields

$$(1) \quad u = (u^1, u^2, u^3) = u(t, x^1, x^2, x^3) = u(t, x)$$

which satisfy the system of equations

$$(2) \quad u_t^\alpha + p_{, \alpha} = -u^\beta u_{, \beta}^\alpha + \nu u_{, \beta}^\alpha p_{, \beta} + f^\alpha; \quad (\alpha=1, 2, 3);$$

$$(3) \quad u_{, \alpha}^\alpha = 0.$$

Here p is the scalar pressure, ν is the kinematic viscosity coefficient, $f^\alpha(t, x)$ is an arbitrary prescribed driving force which is defined for $t \geq 0$, and the density is assumed to be unity. We employ the summation convention over repeated indices. (2) is the system of (forced) Navier-Stokes equations, and (3) is the condition of incompressibility. We assume that the solution u is uniquely determined for $t \geq 0$ by specifying initial conditions at $t = 0$, and appropriate boundary conditions at spatial infinity. Unless explicitly indicated, integrations with respect to t will be over the semi-infinite interval $0 \leq t$. Integrations with respect to $x = (x^1, x^2, x^3)$ and $k = (k^1, k^2, k^3)$ will always be over the entire 3-dimensional space.

For functionals $F[u]$ (which assign a numerical value to each vector field u) we define the mean value

$$(4) \quad \langle F \rangle = \int F[u] dP.$$

The integration here is over the entire space of velocity vector fields which satisfy (1), (3) and the appropriate boundary condition. The probability is non-negative and normalized, i.e., $\int_A dP$ is non-negative if A is an arbitrary subset of the space, and is equal to one if A is the whole space. It follows that the probability distribution cannot assign a negative probability to any velocity field.

Now let

$$(5) \quad y = (y^1, y^2, y^3) = y(t, x)$$

be an arbitrary real vector field which vanishes at spatial infinity. Let

$$(6) \quad [y, u] = \int y^{\alpha} u^{\alpha} dt dx ,$$

and

$$(7) \quad \Gamma[y] = \langle e^{i[y, u]} \rangle = \int e^{i[y, u]} dP .$$

Then Γ is a functional of y and will be called the characteristic functional of P . By functional differentiation of (7) we obtain

$$(8) \quad \frac{\delta^n \Gamma}{\delta y^{\alpha_1}(t_1, x_1) \dots \delta y^{\alpha_n}(t_n, x_n)} = \langle i^{n-1} u^{\alpha_1}(t_1, x_1) \dots u^{\alpha_n}(t_n, x_n) e^{i[y, u]} \rangle ;$$

hence

$$(ii) \quad \langle u^{(1)}(t_1, x_1) \dots u^{(n)}(t_n, x_n) \rangle = i^{-n} \frac{\langle u^{(n)}(t_1, x_1) \dots u^{(n)}(t_n, x_n) \rangle}{\langle u^{(1)}(t_1, x_1) \dots u^{(1)}(t_n, x_n) \rangle} \quad (10)$$

The left side of (10) is just the n -point space-time correlation function of the velocity field.

An alternative formulation of the statistical theory of turbulence may be based on a sequence of probability distributions p in n -dimensional space, $n=1, 2, \dots$. These distributions are defined for n arbitrary space-time points $(t_1, x_1), \dots, (t_n, x_n)$ and n arbitrary components $u^{(1)}, \dots, u^{(n)}$ of the vector field u ; $p(\xi_1, \dots, \xi_n | \xi_1, \dots, \xi_n)$ being the joint probability that all $u^{(j)}(t_j, x_j)$ are in the interval $(\xi_j, \xi_j + d\xi_j)$; $j=1, \dots, n$. If we introduce the characteristic function

$$(11) \quad \gamma(\tau_1, \dots, \tau_n) = \int \exp \left\{ i \sum_j \tau_j \xi_j \right\} p(\xi_1, \dots, \xi_n | \xi_1, \dots, \xi_n)$$

of this probability distribution, then

$$(12) \quad p(\xi_1, \dots, \xi_n) = \langle \pi \rangle^{-n} \int \exp \left\{ -i \sum_j \tau_j \xi_j \right\} \gamma(\tau_1, \dots, \tau_n | \xi_1, \dots, \xi_n).$$

Now let us evaluate the characteristic functional Γ for the special argument function y whose components are given by

$$(13) \quad y^{(j)}(t, x) = \sum_{j=1}^n \tau_j u_j^{(j)} \delta(t - t_j) \delta(x - x_j) \quad .$$

Here $\delta_{\alpha\beta}$ is the Krönecker delta, and $\delta(t)$ and $\delta(\pi)$ are the one-dimensional and three-dimensional Dirac delta functions. For y given by (12),

$$(13) \quad [y, u] = \sum_{j=1}^n \sigma_j u^{\alpha_j}(t_j, x_j),$$

$$(14) \quad \Gamma[y] = \langle \exp \left\{ i \sum_{j=1}^n \sigma_j u^{\alpha_j}(t_j, x_j) \right\} \rangle = \Gamma(\sigma_1, \dots, \sigma_n),$$

and

$$(15) \quad p(\xi_1, \dots, \xi_n) = (2\pi)^{-n} \int \exp \left\{ -i \sum_j \sigma_j \xi_j \right\} \Gamma[y] d\sigma_1, \dots, d\sigma_n.$$

Thus we see that the probability distribution functions p and their characteristic functions Γ can be obtained from the characteristic functional $\Gamma[y]$ by means of the special choice (12) of y .

3. Derivation of the functional differential equation.

In this section, we obtain equations for Γ from the basic flow equations (2) and (3). To do this we first note that

$$(16) \quad \frac{\delta \Gamma}{\delta y^{\alpha}(t, x)} = \langle i u^{\alpha}(t, x) e^{i[y, u]} \rangle; \quad \frac{\delta^2 \Gamma}{\delta y^{\alpha}(t, x) \delta y^{\beta}(t, x)} = - \langle u^{\alpha}(t, x) u^{\beta}(t, x) e^{i[y, u]} \rangle.$$

Hence, from (2),

$$(17) \quad \frac{\partial}{\partial t} \frac{\delta \Gamma}{\delta y^\alpha(t, x)} = \langle i u_t^\alpha e^{i[y, u]} \rangle = \langle i [-p_\alpha - u_{x^\beta}^\beta + u_{x^\beta}^\alpha + f^\alpha] e^{i[y, u]} \rangle ;$$

$$\alpha = 1, 2, 3 .$$

Now,

$$(18) \quad \frac{\partial^2}{\partial x^\beta \partial x^\beta} \frac{\delta \Gamma}{\delta y^\alpha(t, x)} = \langle i u_{x^\beta x^\beta}^\alpha e^{i[y, u]} \rangle ,$$

and

$$(19) \quad \frac{\partial}{\partial x^\beta} \frac{\delta^2 \Gamma}{\delta y^\alpha(t, x) \delta y^\beta(t, x)} = - \langle [u_{x^\beta}^\alpha u^\beta + u_{x^\beta}^\alpha u^\beta] e^{i[y, u]} \rangle .$$

But the quantity $u_{x^\beta}^\beta$ in (19) vanishes by virtue of (3), hence the last three equations yield

$$(20) \quad \frac{\partial}{\partial t} \frac{\delta \Gamma}{\delta y^\alpha(t, x)} = i \frac{\partial}{\partial x^\beta} \frac{\delta^2 \Gamma}{\delta y^\alpha(t, x) \delta y^\beta(t, x)} + v \frac{\partial^2}{\partial x^\beta \partial x^\beta} \frac{\delta \Gamma}{\delta y^\alpha(t, x)} + i f^\alpha T - \frac{\partial \Pi}{\partial x^\alpha} ;$$

$$\alpha = 1, 2, 3 ;$$

where

$$\Pi = \langle i p(t, x) e^{i[y, u]} \rangle .$$

Various devices for eliminating the pressure term $-\frac{\partial \Pi}{\partial x^\alpha}$ from (20)

are possible. The method we shall use here is to introduce the "testing field"

$$\eta(t, x) = (\eta^1, \eta^2, \eta^3)$$

which vanishes sufficiently rapidly at spatial infinity and satisfies the condition

$$(22) \quad \eta_{\alpha}^{\alpha} = 0.$$

Then

$$(23) \quad \int \eta^{\alpha} \frac{\partial \Pi}{\partial x^{\alpha}} dx = - \int \eta_{\alpha}^{\alpha} \Pi dx = 0,$$

and from (20) we obtain

$$(24) \quad \int \eta^{\alpha}(t, x) \left\{ \frac{\partial}{\partial t} \frac{\delta \Gamma}{\delta y^{\alpha}(t, x)} - i \frac{\partial}{\partial x^{\beta}} \frac{\delta^2 \Gamma}{\delta y^{\alpha}(t, x) \delta y^{\beta}(t, x)} - \nu \frac{\partial^2}{\partial x^{\beta} \partial x^{\beta}} \frac{\delta \Gamma}{\delta y^{\alpha}(t, x)} - i f^{\alpha} \Gamma \right\} dt dx = 0.$$

This is a functional differential equation for $\Gamma[y]$. It must be satisfied by Γ for all testing fields η which satisfy (22). An alternative to the introduction of the testing field η is the elimination of the pressure from the Navier-Stokes equation at the outset. By using the boundary conditions and the incompressibility condition the pressure field may be expressed as the sum of a second-degree polynomial functional of the velocity field and a linear functional of the force field. (Cf. [4], for example.)

In addition to (24) which now replaces the Navier-Stokes equations (2),

we have the following equation which replaces the incompressibility condition (3):

$$(25) \quad \frac{\partial}{\partial x^\alpha} \frac{\delta \Gamma}{\delta y^\alpha(t, x)} = 0.$$

This is an immediate consequence of (3) and (16).

Three further conditions on Γ are

$$(26) \quad \Gamma[0] = 1, \quad \Gamma^*[y] = \Gamma[-y]; \quad |\Gamma[y]| \leq 1.$$

Here $*$ denotes the complex conjugate. The first two of these conditions follow immediately from (7) and the normalization of P . The third condition is a consequence of the normalization and non-negativity of P . This condition is the simplest of an infinite sequence of realizability inequalities which Γ must satisfy. The entire set of inequalities may be summarized by the statement that $\Gamma[y]$ must have an everywhere non-negative functional Fourier transform. The realizability inequalities imply corresponding inequalities for the velocity moments. [5]

4. Transformation to k-space.

A formal simplification of the functional differential equation (24) can be obtained by Fourier transformation. Let

$$(27) \quad z^\alpha(t, k) = \int y^\alpha(t, x) e^{-ik \cdot x} dx; \quad y^\alpha(t, x) = (2\pi)^{-3} \int z^\alpha(t, k) e^{ik \cdot x} dk;$$

$$(28) \quad \eta^{\alpha}(t, k) = \int \eta^{\alpha}(t, x) e^{-ik \cdot x} dx ; \quad \eta^{\alpha}(t, x) = (2\pi)^{-3} \int \eta^{\alpha}(t, k) e^{ik \cdot x} dk ;$$

$$(29) \quad g^{\alpha}(t, k) = (2\pi)^{-3} \int f^{\alpha}(t, x) e^{ik \cdot x} dx ; \quad f^{\alpha}(t, x) = \int g^{\alpha}(t, k) e^{-ik \cdot x} dk .$$

Here $\alpha = 1, 2, 3$ and $k \cdot x = k^{\alpha} x^{\alpha}$. Since y and f are real, it follows from (27) that

$$(30) \quad z^{*}(t, k) = z(t, -k) ; \quad g^{*}(t, k) = g(t, -k).$$

An immediate consequence of (22) is the condition

$$(31) \quad k^{\alpha} \lambda^{\alpha} = 0 .$$

We now set

$$(32) \quad \Gamma_1[z] = \Gamma[y] = \Gamma[(2\pi)^{-3} \int z(t, k) e^{ik \cdot x} dk] .$$

In what follows, we shall omit the subscript 1, since it is always clear from the argument function whether Γ_1 or Γ is intended. By functional differentiation, we obtain

$$(33) \quad \frac{\delta^n \Gamma}{\delta y^{\alpha_1}(t_1, x_1) \dots \delta y^{\alpha_n}(t_n, x_n)} = \int \frac{\delta^n \Gamma}{\delta z^{\alpha_1}(t_1, k_1) \dots \delta z^{\alpha_n}(t_n, k_n)} \exp \left\{ -i(x_1 \cdot k_1 + \dots + x_n \cdot k_n) \right\} dk_1 \dots dk_n .$$

In particular

$$(34) \quad \frac{\delta \Gamma}{\delta y^{\alpha}(t, x)} = \int \frac{\delta \Gamma}{\delta z^{\alpha}(t, k)} e^{-ix \cdot k} dk ;$$

$$\frac{\delta^2 \Gamma}{\delta y^{\alpha}(t, x) \delta y^{\beta}(t, x)} = \int \frac{\delta^2 \Gamma}{\delta z^{\alpha}(t, k_1) \delta t^{\beta}(t, k_2)} e^{-ix \cdot (k_1 + k_2)} dk_1 dk_2 .$$

Now each term of (24) may be transformed to the k-representation. Thus

$$(35) \quad \int \eta^{\alpha}(t, x) \frac{\partial}{\partial t} \frac{\delta \Gamma}{\delta y^{\alpha}(t, x)} dt dx = (2\pi)^{-3} \int \lambda^{\alpha}(t, k') e^{ik' \cdot x} \frac{\partial}{\partial t} \frac{\delta \Gamma}{\delta z^{\alpha}(t, k)} e^{-ik \cdot x} dk' dk dt dx$$

$$= \int \lambda^{\alpha}(t, k) \frac{\partial}{\partial t} \frac{\delta \Gamma}{\delta z^{\alpha}(t, k)} dt dk ;$$

$$(36) \quad -\nu \int \eta^{\alpha}(t, x) \frac{\partial^2}{\partial x^{\beta} \partial x^{\beta}} \frac{\delta \Gamma}{\delta y^{\alpha}(t, x)} dt dx$$

$$= \nu (2\pi)^{-3} \int \lambda^{\alpha}(t, k') e^{ik' \cdot x} k^{\beta} k^{\beta} \frac{\delta \Gamma}{\delta z^{\alpha}(t, k)} e^{-ik \cdot x} dk' dk dt dx = \nu \int \lambda^{\alpha}(t, k) k^2 \frac{\delta \Gamma}{\delta z^{\alpha}(t, k)} dt dk ;$$

$$\begin{aligned}
 (37) \quad & -i \int \eta^\alpha(t, x) \frac{\partial}{\partial x^\beta} \frac{\delta^2 \Gamma}{\delta y^\alpha(t, x) \delta y^\beta(t, x)} dt dx \\
 & = \frac{(-i)^2}{(2\pi)^3} \int \lambda^\alpha(t, k) e^{ik \cdot x} (k_1^\beta + k_2^\beta) \frac{\delta^2 \Gamma}{\delta z^\alpha(t, k_1) \delta z^\beta(t, k_2)} e^{-ix \cdot (k_1 + k_2)} dk_1 dk_2 dk dt dx \\
 & = - \int \lambda^\alpha(t, k_1 + k_2) (k_1^\beta + k_2^\beta) \frac{\delta^2 \Gamma}{\delta z^\alpha(t, k_1) \delta z^\beta(t, k_2)} dk_1 dk_2 dt ;
 \end{aligned}$$

and

$$\begin{aligned}
 (38) \quad & -i \int \eta^\alpha f^\alpha \Gamma dt dx = \frac{-i\Gamma}{(2\pi)^3} \int \lambda^\alpha(t, k') e^{ik' \cdot x} g^\alpha(t, k) e^{-ik \cdot x} dk' dk dx dt \\
 & = -i\Gamma \int \lambda^\alpha(t, k) g^\alpha(t, k) dt dk = -i\Gamma \int \lambda^\alpha(t, k) [\delta_{\alpha\beta} - k^{-2} k^\alpha k^\beta] g^\beta(t, k) dt dk .
 \end{aligned}$$

The last equation is a consequence of (31). If we insert (35-38) in (24) we obtain

$$\begin{aligned}
 (39) \quad & \int \lambda^\alpha(t, k) \left\{ \left[\frac{\partial}{\partial t} + \nu k^2 \right] \frac{\delta \Gamma}{\delta z^\alpha(t, k)} - i [\delta_{\alpha\beta} - k^{-2} k^\alpha k^\beta] g^\beta(t, k) \Gamma \right\} dt dk \\
 & - \int \lambda^\alpha(t, k_1 + k_2) (k_1^\beta + k_2^\beta) \frac{\delta^2 \Gamma}{\delta z^\alpha(t, k_1) \delta z^\beta(t, k_2)} dk_1 dk_2 dt = 0 .
 \end{aligned}$$

This is the functional differential equation for $\Gamma[z]$. It must be satisfied by $\Gamma[z]$ for all testing fields λ which satisfy (31). In the k -space representation, the incompressibility condition (25) becomes

$$(40) \quad k^\alpha \frac{\delta \Gamma}{\delta z^\alpha(t, k)} = 0,$$

and from (26) we have the three conditions

$$(41) \quad \Gamma[0] = 1, \quad \Gamma^*[z] = \Gamma[-z]; \quad |\Gamma(z)| \leq 1.$$

Our explicit introduction of the projection operator $\delta_{\alpha\beta} - k^{-2} k_\alpha k_\beta$ into (32) emphasizes the fact that the velocity field can respond only to the transverse (i.e., solenoidal) part of the driving forces.

A formal representation for $\Gamma[z]$ is the "Taylor expansion",

$$(42) \quad \Gamma[z] = P_0 + \sum_{n=1}^{\infty} \frac{1}{n!} \int P_{\alpha_1, \dots, \alpha_n}(t_1, k_1, \dots, t_n, k_n) \times \prod_{j=1}^n z^{\alpha_j}(t_j, k_j) dt_j dk_1 \dots dt_n dk_n,$$

where

$$(43) \quad P_{\alpha_1, \dots, \alpha_n}(t_1, k_1, \dots, t_n, k_n) = \left. \frac{\delta^n \Gamma}{\delta z^{\alpha_1}(t_1, k_1), \dots, \delta z^{\alpha_n}(t_n, k_n)} \right|_{z=0}.$$

Now from (40) we see that

$$(44) \quad \partial_j^j P_{\alpha_1, \dots, \alpha_n} = 0, \quad j=1, \dots, n;$$

and from this it follows that there exist functions $Q_{\beta_1, \dots, \beta_n}$ such that

$$(45) \quad P_{\alpha_1, \dots, \alpha_n}(t_1, k_1, \dots, t_n, k_n) = \prod_{\nu=1}^n \left[\delta_{\alpha_\nu \beta_\nu} - k_\nu^{-2} k_\nu^{\alpha_\nu \beta_\nu} \right] Q_{\beta_1, \dots, \beta_n}(t_1, k_1, \dots, t_n, k_n)$$

From (41), we have $P_0 = 1$ so that Γ may be represented in the form

$$(46) \quad \Gamma[z] = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int \prod_{\nu=1}^n \left[\delta_{\alpha_\nu \beta_\nu} - k_\nu^{-2} k_\nu^{\alpha_\nu \beta_\nu} \right] Q_{\beta_1, \dots, \beta_n} \prod_{j=1}^n z^{\alpha_j} dt_1 dk_1 \dots dt_n dk_n.$$

The incompressibility condition (40) now is automatically satisfied.

From (9) and (33) we see that the space-time correlation function

$$(47) \quad U_{\alpha_1, \dots, \alpha_n}(t_1, x_1, \dots, t_n, x_n) = \langle u^{\alpha_1}(t_1, x_1) \dots u^{\alpha_n}(t_n, x_n) \rangle$$

is given by

$$(48) \quad U_{\alpha_1, \dots, \alpha_n}(t_1, x_1, \dots, t_n, x_n) = i^{-n} \int P_{\alpha_1, \dots, \alpha_n}(t_1, k_1, \dots, t_n, k_n) e^{-i(x_1 \cdot k_1 + \dots + x_n \cdot k_n)} dk_1 \dots dk_n.$$

For many purposes it is useful to transform (39) so that a characteristic Reynolds number appears rather than the viscosity coefficient ν . This can be done by introducing some appropriate characteristic correlation length r_0 and characteristic velocity v_0 . We then measure lengths and velocities in units r_0 and v_0 and time in the unit $\frac{r_0^2}{v}$. This last unit is the characteristic time

for viscous decay of a velocity field whose scale of spatial variation is r_0 . If we denote quantities measured in the original units by primed symbols and corresponding quantities measured in the new units by un-primed symbols, then

$$(49) \quad t' = \frac{r_0^2}{\nu} t, \quad x' = r_0 x, \quad u' = v_0 u, \quad k' = \frac{1}{r_0} k, \quad f' = \frac{v_0 \nu}{r_0^2} f,$$

$$y^{\alpha'} dx' = \frac{\nu}{v_0 r_0^2} y^{\alpha} dx, \quad z^{\alpha'} = \frac{\nu}{v_0 r_0^2} z^{\alpha}, \quad \text{and} \quad g^{\alpha'} dk' dt' = v_0 g^{\alpha} dk dt.$$

From (42) and (43) we see that

$$\frac{\delta^n \Gamma}{\delta z_1^{\alpha_1} \dots \delta z_n^{\alpha_n}} \prod_{j=1}^n z^{\alpha_j}(t_j, k_j) dt_1 dk_1 \dots dt_n dk_n$$

is dimensionless, and a simple calculation shows that in terms of the new units (39) becomes

$$(50) \quad \int \lambda^{\alpha}(t, k) \left\{ \left[\frac{\partial}{\partial t} + k^2 \right] \frac{\delta \Gamma}{\delta z^{\alpha}(t, k)} - i \left[\delta_{\alpha\beta} - k^{-2} k^{\alpha} k^{\beta} \right] g^{\beta}(t, k) \Gamma \right\} dt dk$$

$$- R \int \lambda^{\alpha}(t, k_1 + k_2) (k_1^{\beta} + k_2^{\beta}) \frac{\delta^2 \Gamma}{\delta z^{\alpha}(t, k_1) \delta z^{\beta}(t, k_2)} dk_1 dk_2 dt = 0.$$

Here the Reynolds number R is given by

$$(51) \quad R = \frac{v_0 r_0}{\nu}.$$

5. Solution of the functional differential equation for zero Reynolds number

If we take the limit $R \rightarrow 0$, (50) reduces to

$$(52) \quad \int \lambda^\alpha(t, k) \left\{ \left[\frac{\partial}{\partial t} + k^2 \right] \frac{\delta \Gamma}{\delta z^\alpha(t, k)} - i \left[\delta_{\alpha\beta} - k^{-2} k^\alpha k^\beta \right] g^\beta(t, k) \Gamma \right\} dt dk = 0.$$

In this case, we can obtain a solution of the functional differential equation and recover the results of the theory of "weak turbulence", derived by Reissner and by Batchelor and Townsend.^[1] We begin with the following

Lemma. If $\int \lambda^\alpha(t, k) q^\alpha(t, k) dt dk = 0$ for all λ which satisfy (31), then there exists a scalar function $p(t, k)$ such that

$$(53) \quad q^\alpha(t, k) = k^\alpha p(t, k); \quad \alpha = 1, 2, 3.$$

Proof:

For arbitrary $\tau(t, k) = (\tau^1, \tau^2, \tau^3)$, set $\lambda^\alpha = \tau^\alpha - k^{-2} k^\beta \tau^\beta k^\alpha$.

Then $\lambda^\alpha k^\alpha = 0$ and

$$\int \tau^\alpha (q^\alpha - k^{-2} k^\beta q^\beta k^\alpha) dt dk = \int (\tau^\alpha - k^{-2} k^\beta \tau^\beta k^\alpha) q^\alpha dt dk = \int \lambda^\alpha q^\alpha dt dk = 0.$$

Hence, since τ is arbitrary, $q^\alpha = k^\alpha (k^{-2} k^\beta q^\beta)$; $\alpha = 1, 2, 3$.

If we apply the lemma to (52), we obtain

$$(54) \quad \left[\frac{\partial}{\partial t} + k^2 \right] \frac{\delta \Gamma}{\delta z^\alpha(t, k)} - i \left[\delta_{\alpha\beta} - k^{-2} k^\alpha k^\beta \right] g^\beta(t, k) \Gamma = p k^\alpha; \quad \alpha = 1, 2, 3.$$

However, from (40) and (54) we see that $pk^{\alpha\omega} = pk^2 = 0$. Hence $p=0$ and we obtain

$$(55) \quad \left[\frac{\partial}{\partial t} + k^2 \right] \frac{\delta \Gamma}{\delta z^\alpha(t,k)} - i \left[\delta_{\alpha\beta} - k^{-2} k^\alpha k^\beta \right] g^\beta(t,k) \Gamma = 0; \quad \alpha=1,2,3.$$

From (55) we see that $\frac{\delta \Gamma}{\delta z^\alpha(t,k)}$ satisfies a first order linear inhomogeneous ordinary differential equation in t which is easily solved to yield

$$(56) \quad \frac{\delta \Gamma}{\delta z^\alpha(t,k)} = i h^\alpha(t,k) \Gamma + e^{-k^2 t} B^\alpha[z,k].$$

Here

$$(57) \quad h^\alpha(t,k) = (\delta_{\alpha\beta} - k^{-2} k^\alpha k^\beta) e^{-k^2 t} \int_0^t g^\beta(\tau,k) e^{k^2 \tau} d\tau,$$

and from (40) it follows that

$$(58) \quad k^\alpha B^\alpha = 0.$$

Now the substitution

$$(59) \quad \Gamma[z] = V[z] \exp \left\{ i \int h^\alpha(t,k) z^\alpha(t,k) dt dk \right\}$$

is suggested by the form of (56). Indeed (59) is the analogue of the usual exponential form of solution of a linear ordinary differential equation, and the functional $V[z]$ plays the role of the coefficient function in the method of "variation of parameters". By functional differentiation (59) yields

$$(60) \quad \frac{\delta \Gamma}{z^\alpha(t, k)} = i h^\alpha(t, k) \Gamma + \exp \left\{ i \int h^\alpha z^\alpha dt dk \right\} \frac{\delta V}{\delta z^\alpha(t, k)} ,$$

and by comparing (56) and (60) we see that

$$(61) \quad \frac{\delta V}{\delta z^\alpha(t, k)} = e^{-k^2 t} C^\alpha(z, k) ,$$

where

$$(62) \quad C^\alpha = \exp \left\{ -i \int h^\beta z^\beta dt dk \right\} B^\alpha .$$

From (62) and (58) it follows that

$$(63) \quad k^\alpha C^\alpha = 0 .$$

The functional C^α is arbitrary so far, except for the condition (63) and the condition

$$(64) \quad e^{-k_1^2 t_1} \frac{\delta C^\alpha(z, k_1)}{\delta z^\beta(t_2, k_2)} = e^{-k_2^2 t_2} \frac{\delta C^\beta(z, k_2)}{\delta z^\alpha(t_1, k_1)} ,$$

which follows from the fact that

$$(65) \quad \frac{\delta^2 V}{\delta z^\alpha(t_1, k_1) \delta z^\beta(t_2, k_2)} = \frac{\delta^2 V}{\delta z^\beta(t_2, k_2) \delta z^\alpha(t_1, k_1)} .$$

Now equation (61) may be solved for $V[z]$ in several ways. One way is to expand V and C^α in "Taylor series" and compare the coefficients. It is easily seen that the solution may be expressed in the form

$$(66) \quad V[z] = A[w]; \quad w^\beta(k) = \int_0^\infty e^{-k^2\tau} (\delta_{\alpha\beta} - k^{-2} k^\alpha k^\beta) z^\alpha(\tau, k) d\tau,$$

because

$$(67) \quad \frac{\delta w^\beta(k)}{\delta z^\alpha(t, k)} = \int_0^\infty e^{-k^2\tau} (\delta_{\alpha\beta} - k^{-2} k^\alpha k^\beta) \delta(\tau - t) \delta(k - k) d\tau \\ = e^{-k^2t} (\delta_{\alpha\beta} - k^{-2} k^\alpha k^\beta) \delta(k - k),$$

and for any choice of the functional $A[w]$,

$$(68) \quad \frac{\delta V}{\delta z^\alpha(t, k)} = \int \frac{\delta A}{\delta w^\beta(k)} \frac{\delta w^\beta(k)}{\delta z^\alpha(t, k)} dk = e^{-k^2t} (\delta_{\alpha\beta} - k^{-2} k^\alpha k^\beta) \frac{\delta A}{\delta w^\beta(k)}.$$

Thus (61), (63) and (64) are satisfied by (66).

We now insert (66) in (59) and obtain the solution

$$(69) \quad \Gamma[z] = A[w] \exp \left\{ i \int h^\alpha(t, k) z^\alpha(t, k) dt dk \right\}; \\ h^\alpha(t, k) = (\delta_{\alpha\beta} - k^{-2} k^\alpha k^\beta) e^{-k^2t} \int_0^t g^\beta(\tau, k) e^{k^2\tau} d\tau;$$

$$w^{\alpha}(k) = (\delta_{\alpha\beta} - k^{-2} k^{\alpha} k^{\beta}) \int_0^{\infty} z^{\beta}(\tau, k) e^{-k^2 \tau} d\tau.$$

The functional A is, so far, arbitrary except for the conditions

$$(70) \quad A[0] = 1, \quad A^*[w] = A[-w], \quad |A(w)| \leq 1$$

which follow from (41) and (30). It is easy to see that the $\Gamma[z]$ given by (69) satisfies (52) and (40).

It is not difficult to show that the functional A is simply the characteristic functional (in the k-representation) of the distribution of initial velocity fields (cf. Section 7). This distribution must be non-negative, a fact which implies restrictions on the possible forms of A.

For the case of zero driving force, $f^{\alpha} \equiv g^{\alpha} \equiv 0$, the solution (69) reduces to

$$(71) \quad \Gamma[z] = A[w].$$

By functional differentiation we obtain

$$(72) \quad \frac{\delta^n \Gamma}{\delta z_1^{\alpha_1}(t_1, k_1) \dots \delta z_n^{\alpha_n}(t_n, k_n)} = \left[\prod_{\nu=1}^n e^{-k_{\nu}^2 t_{\nu}} (\delta_{\alpha_{\nu} \beta_{\nu}} - k_{\nu}^{-2} k_{\nu}^{\alpha_{\nu}} k_{\nu}^{\beta_{\nu}}) \right] \frac{\delta^n A}{\delta w_1^{\beta_1}(k_1) \dots \delta w_n^{\beta_n}(k_n)},$$

and from (43) and (48) we obtain

$$(73) \quad U_{\alpha_1, \dots, \alpha_n}(t_1, x_1, \dots, t_n, x_n) = i^{-n} \int \left[\prod_{\nu=1}^n e^{-k_{\nu}^2 t_{\nu}} (\delta_{\alpha_{\nu} \beta_{\nu}} - k_{\nu}^{-2} k_{\nu}^{\alpha_{\nu}} k_{\nu}^{\beta_{\nu}}) \right] \\ \times a_{\beta_1, \dots, \beta_n}(k_1, \dots, k_n) \exp \left\{ -i(x_1 \cdot k_1 + \dots + x_n \cdot k_n) \right\} dk_1 \dots dk_n.$$

Here

$$(74) \quad a_{\beta_1, \dots, \beta_n}^{(k_1, \dots, k_n)} = \frac{\delta^n A}{\delta w^{\beta_1}(k_1) \dots \delta w^{\beta_n}(k_n)} \bigg|_{w=0}.$$

Equation (73) provides an explicit formula for the time-dependence of the space-time correlation functions. If, for example, $U_{\alpha_1, \dots, \alpha_n}$ is given for $t_1 = t_1^0, \dots, t_n = t_n^0$, then $a_{\beta_1, \dots, \beta_n}$ can be determined by inverting the resulting Fourier transform.

6. Representation of averaged Green's function

In this section we introduce a sequence of Green's functions which describe the averaged response of the velocity fields to infinitesimal perturbations in the external force field f^α . These functions play an essential role in some recent studies of turbulence.^[4,6] They are defined as follows:

$$(75) \quad G^{\alpha\beta}(t_1, x_1 | t_2, x_2) \equiv \left\langle \frac{\delta u^\alpha(t_1, x_1)}{\delta f^\beta(t_2, x_2)} \right\rangle,$$

$$G^{\alpha\beta\gamma}(t_1, x_1 | t_2, x_2; t_3, x_3) \equiv \left\langle \frac{\delta^2 u^\alpha(t_1, x_1)}{\delta f^\beta(t_2, x_2) \delta f^\gamma(t_3, x_3)} \right\rangle,$$

.

Here $\alpha, \beta, \dots = 1, 2, 3$. If, as we have assumed in section 2, the solutions $u^\alpha(t, x)$ are determined by initial conditions, then the G functions vanish unless $t_1 \geq t_2, t_1 \geq t_3, \dots$; that is, they are retarded Green's functions.

When evaluated at $f = 0$ the Green's functions give the averaged response of freely decaying turbulence to infinitesimal external perturbations.

The Green's functions may be represented as functional derivatives, with respect to y and f , of the solution Γ of (24). Noting (16), we have

$$(75a) \quad G^{\alpha\beta}(t_1, x_1 | t_2, x_2) = i^{-1} \frac{\delta}{\delta f^{\beta}(t_2, x_2)} \left[\frac{\delta \Gamma}{\delta y^{\alpha}(t_1, x_1)} \Big|_{y=0} \right],$$

$$G^{\alpha\beta\gamma}(t_1, x_1 | t_2, x_2; t_3, x_3) = i^{-1} \frac{\delta^2}{\delta f^{\beta}(t_2, x_2) \delta f^{\gamma}(t_3, x_3)} \left[\frac{\delta \Gamma}{\delta y^{\alpha}(t_1, x_1)} \Big|_{y=0} \right],$$

.

In the "weak turbulence" limit, $R \rightarrow 0$, we may use the results of the preceding section to calculate the Green's functions. Only the lowest member $G^{\alpha\beta}$ of the sequence now is non-vanishing. The higher Green's functions describe intrinsically non-linear effects which are ignored in this approximation. In order to calculate $G^{\alpha\beta}$ we combine (69) and (29) to obtain

$$(76) \quad h^{\alpha}(t_1, k_1) = (2\pi)^{-3} (\delta_{\alpha\beta} - k_1^{-2} k_1^{\alpha} k_1^{\beta}) \int_0^{t_1} dt \int dx \exp \left\{ k_1^2 (t - t_1) + i k_1 \cdot x \right\} f^{\beta}(t, x),$$

and functional differentiation of this equation yields

$$\begin{aligned}
 (77) \quad \frac{\delta h^\alpha(t_1, k_1)}{\delta f^\beta(t_2, x_2)} &= (2\pi)^{-3} (\delta_{\alpha\beta} - k_1^{-2} k_1^\alpha k_1^\beta) \int_0^{t_1} dt \int dx \exp \left\{ k_1^2 (t - t_1) + i k_1 \cdot x \right\} \delta(t - t_2) \delta(x - x_2) \\
 &= \begin{cases} (2\pi)^{-3} (\delta_{\alpha\beta} - k_1^{-2} k_1^\alpha k_1^\beta) \exp \left\{ k_1^2 (t_2 - t_1) + i k_1 \cdot x_2 \right\} & \text{if } t_2 \leq t_1 \\ 0 & \text{if } t_2 > t_1 \end{cases} .
 \end{aligned}$$

Now from (69) we see that

$$(78) \quad \left. \frac{\delta \Gamma}{\delta z^\alpha(t_1, k_1)} \right|_{z=0} = i h^\alpha(t_1, k_1) + e^{-k_1^2 t_1} \left(\delta_{\alpha\beta} - k_1^{-2} k_1^\alpha k_1^\beta \right) \left. \frac{\delta A}{\delta w^\beta(k_1)} \right|_{w=0} ,$$

hence

$$(79) \quad \frac{\delta}{\delta f^\beta(t_2, x_2)} \left\{ \left. \frac{\delta \Gamma}{\delta z^\alpha(t_1, k_1)} \right|_{z=0} \right\} = i \frac{\delta h^\alpha(t_1, k_1)}{\delta f^\beta(t_2, x_2)} .$$

It follows that

$$\begin{aligned}
 (80) \quad G^{\alpha\beta}(t_1, x_1 | t_2, x_2) &= i^{-1} \frac{\delta}{\delta f^\beta(t_2, x_2)} \int e^{-i x_1 \cdot k_1} \left\{ \left. \frac{\delta \Gamma}{\delta z^\alpha(t_1, k_1)} \right|_{z=0} \right\} dk_1 \\
 &= \int e^{-i x_1 \cdot k_1} \frac{\delta h^\alpha(t_1, k_1)}{\delta f^\beta(t_2, x_2)} dk_1
 \end{aligned}$$

$$= \begin{cases} (2\pi)^{-3} \int \left(\delta_{\alpha\beta} - k_1^2 k_1^\alpha k_1^\beta \right) \exp \left\{ -k_1^2 (t_1 - t_2) + i k_1 \cdot (x_2 - x_1) \right\} dk_1 & \text{if } t_2 \leq t_1 \\ 0 & \text{if } t_2 > t_1 \end{cases}.$$

7. Treatment of random driving forces

We wish now to describe briefly the generalization of the functional formalism to the case where the driving-force field, as well as the velocity field is statistically distributed. We shall discuss here only the case where the force-field and the initial velocity field are statistically independent. Instead of Γ , we consider the joint characteristic functional

$$(81) \quad \Omega[y, q] = \langle \exp \left\{ i[y, u] + i[q, f] \right\} \rangle,$$

where $y(t, x)$ and $q(t, x)$ are arbitrary vector fields and the averaging $\langle \rangle$ now is over the joint probability distribution of u and f . The characteristic functionals of the initial velocity distribution and of the force field distribution are defined, respectively, by

$$(82) \quad A[v] = \langle \exp \left[i \int v(x) u(0, x) dx \right] \rangle$$

and

$$(83) \quad \psi[q] = \langle e^{i[q, f]} \rangle.$$

Here v is an arbitrary vector function of x . We regard A and ψ as prescribed.

It follows immediately from the statistical independence of force field and initial velocity field that Ω must satisfy the "boundary" condition:

$$(84) \quad \Omega[y, q] = A[v] \psi[q] \quad \text{if } y(t, x) = \delta(t - 0^+)v(x) .$$

In particular,

$$(85) \quad \Omega[0, q] = \psi[q] .$$

All moments of the joint distribution of velocity and forcing fields may be expressed as functional derivatives of Ω . For example, we have

$$(86) \quad \langle f^\alpha(t, x) \rangle = i^{-1} \frac{\delta \Omega}{\delta q^\alpha(t, x)} \Big|_{y, q=0} ,$$

$$\langle u^\alpha(t, x) f^\beta(t', x') \rangle = i^{-2} \frac{\delta^2 \Omega}{\delta y^\alpha(t, x) \delta q^\beta(t', x')} \Big|_{y, q=0} .$$

If now we retrace the derivation of (24) and (25), we find

$$(87) \quad \int \eta^\alpha(t, x) \left\{ \left(\frac{\partial}{\partial t} - v \frac{\partial^2}{\partial x^\beta \partial x^\beta} \right) \frac{\delta \Omega}{\delta y^\alpha(t, x)} \right. \\ \left. - i \frac{\partial}{\partial x^\beta} \frac{\delta^2 \Omega}{\delta y^\alpha(t, x) \delta y^\beta(t, x)} - \frac{\delta \Omega}{\delta q^\alpha(t, x)} \right\} dt dx = 0$$

and

$$(88) \quad \frac{\partial}{\partial x^\alpha} \frac{\delta \Omega}{\delta y^\alpha(t, x)} = 0 .$$

It is desired to find the solution of (87) and (88) which satisfies (84).

In order to illustrate the formalism, we shall give the solution in the limiting case of vanishing Reynolds number. (We choose to retain the x-space representation here, rather than make the transformation to the equivalent k-space formalism.) In the zero Reynolds number limit, (87) becomes

$$(89) \quad \int \eta^\alpha(t, x) \left\{ \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^\beta \partial x^\beta} \right) \frac{\delta \Omega}{\delta y^\alpha(t, x)} - \frac{\delta \Omega}{\delta q^\alpha(t, x)} \right\} dt \, dx = 0 .$$

where the units introduced in (49) are used. The solution of (89) which satisfies (84) and (88) is

$$(90) \quad \Omega[y, q] = A[\hat{r}] \, \psi[q+r] ,$$

where

$$(91) \quad \hat{r}^\alpha(x) \equiv r^\alpha(0, x)$$

and

$$(92) \quad r^\alpha(t, x) \equiv \int y^\beta(t', x') G^{\beta\alpha}(t', x' | t, x) dt' \, dx' .$$

The validity of (90) may be verified by substitution and use of the definitions (82) and (83). We shall only outline the demonstration here. Equation (84), first of all, follows directly from the expression (80) for the Green's function. The various functional derivatives which occur in (88) and (89) may be expressed in terms of f and G by use of the chain rule for functional differentiation. Equation (88) then follows immediately from (80). Equation (89) follows from (80) and (22), after an appropriate partial integration is performed.

8. Conditions for homogeneity

We wish finally to see how the special conditions of homogeneous turbulence appear in the present formalism. Homogeneity is defined in terms of the probability distribution P introduced in section 2. P is said to be homogeneous if it is invariant under all translations L_a ; $a = (a^1, a^2, a^3)$; where

$$(93) \quad L_a u(t, x) = u(t, x + a).$$

From (6) and (7) we see that P is homogeneous if and only if

$$(94) \quad \Gamma[L_a y] = \Gamma[y], \quad \text{for all } a;$$

and, of course, this condition is equivalent to the condition

$$(95) \quad \Gamma[L_a z] = \Gamma[z], \quad \text{for all } a;$$

where

$$(96) \quad L_a z(t, k) = \int y(t, x + a) e^{-ik \cdot x} dx = \int y(t, x) e^{-ik \cdot (x - a)} = e^{ia \cdot k} z(t, k).$$

From (46) we see that

$$(97) \quad \Gamma[L_a z] = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int \prod_{\nu=1}^n \left[\delta_{\alpha_{\nu} \beta_{\nu}} - k_{\nu}^2 k_{\nu}^{\alpha_{\nu}} k_{\nu}^{\beta_{\nu}} \right] Q_{\beta_1, \dots, \beta_n} \prod_{j=1}^n e^{ia \cdot k_j} z^{\alpha_j} dt_1 dk_1 \dots dt_n dk_n.$$

Thus (95) implies that for all a and $n=1, 2, \dots$

$$(98) \quad Q_{\beta_1, \dots, \beta_n}(t_1, k_1, \dots, t_n, k_n) = \exp \left\{ ia \cdot (k_1 + \dots + k_n) \right\} Q_{\beta_1, \dots, \beta_n}(t_1, k_1, \dots, t_n, k_n) .$$

It follows that another equivalent condition for homogeneity is that there exist functions $H_{\beta_1, \dots, \beta_n}(t_1, k_1, \dots, t_n, k_n)$ such that

$$(99) \quad Q_{\beta_1, \dots, \beta_n}(t_1, k_1, \dots, t_n, k_n) = \delta(k_1 + \dots + k_n) H_{\beta_1, \dots, \beta_n}(t_1, k_1, \dots, t_n, k_n) .$$

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